Wild polynomial automorphisms over a field of positive characteristic

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Let k be a field, $k[x_1, \dots, x_n]$ the polynomial ring in n variables over k, and ϕ an automorphism of the k-algebra $k[x_1, \dots, x_n]$. We say that ϕ is *affine* if $\phi(x_1), \dots, \phi(x_n)$ are linear polynomials, and *elementary* if there exists $1 \leq l \leq n$ such that

$$\begin{aligned} &\phi(x_l) \in x_l + k[x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n] \\ &\phi(x_i) = x_i \quad \text{for all} \quad i \neq l. \end{aligned}$$

We say that ϕ is *tame* if ϕ is obtained by composing affine automorphisms and elementary automorphisms, and *wild* otherwise.

Around 1950, Jung [1] and van der Kulk [2] showed that every automorphism of $k[x_1, x_2]$ is tame. In 1972, Nagata [4] conjectured that there exists a wild automorphism of $k[x_1, x_2, x_3]$, and gave the following automorphism as a candidate wild automorphism:

$$\begin{aligned} &x_1 \mapsto x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2 x_3 \\ &x_2 \mapsto x_2 + (x_1x_3 + x_2^2)x_3 \\ &x_x \mapsto x_3. \end{aligned}$$

In 2003, Shestakov-Umirbaev [5, 6] settled this conjecture in the affirmative for $\operatorname{char} k = 0$. Therefore, a major remaining question is whether a wild automorphism exists when $\operatorname{char} k > 0$.

Recently, I proved that Nagata's conjecture is true if $\operatorname{char} k \geq 7$. This is the first time that the existence of a wild automorphism has been confirmed in the positive characteristic case. In this talk, I will explain the proof strategy and the key inequality used in the proof.

References

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